An improved version of the 4×4 'transfer matrix' method for periodic magnetic multilayers

1989 J. Phys.: Condens. Matter 1 7977

(http://iopscience.iop.org/0953-8984/1/42/021)

J. Phys.: Condens. Matter 1 (1989) 7977–7981. Printed in the UK

LETTER TO THE EDITOR

An improved version of the 4 × 4 'transfer matrix' method for periodic magnetic multilayers

Zhu Ninjia and Cao Shu
Department of Physics, Southeast University, Nanjing, People's Republic of China

Received 10 April 1989

Abstract. An improved version of the 4 × 4 'transfer matrix' method is discussed. This method for periodic magnetic multilayers is used to deduce the dispersion relations for magnetic multilayers with the elementary unit consisting of N different magnetic layers in rectangular metallic waveguides, and it proves to be possible to distinguish the degenerate states of different eigenvalues in some cases.

In recent years, the interactions within the spin wave excitations (magnons) of condensed media and between these and the electromagnetic wave in periodic magnetic multilayers (PPM) have attracted much attention, both theoretical and experimental [1–10]. As regards theory, many researchers have undertaken not only to give a good description of the collective behaviour of magnetic excitations in PMM using the 'transfer matrix' (T-matrix) method [2, 5, 9, 11, 12], but also to widen the application of the method (this was very clearly, and elegantly, discussed recently by Barnaś [11, 12]) to the theoretical analysis of semi-infinite multilayers (their periodicity is destroyed).

But for problems previously addressed [2, 5, 9, 11, 12] the system of infinite or semi-infinite multilayers without limited boundaries involves only two independent equations for the boundary conditions in the interfaces of two different materials, so the T-matrix is in 2 × 2 form. For the magnetic multilayer properties in metallic waveguides, however, there are four independent equations for the boundary conditions, so the T-matrix takes on a 4 × 4 form. In this Letter we present theoretical discussions of the 4 × 4 T-matrix method in a general form and its application in the calculation of magnetic polaritons of PMM with the elementary unit consisting of N different magnetic layers in the rectangular metallic waveguides.

In 1988, Barnaś [12] derived the general dispersion equations for exchange, magnetostatic and retarded waves in infinite and semi-infinite magnetic multilayers with the aid of the T-matrix method. Because of the interfacial characteristics of the above problems, there are only two equations for the boundary conditions. The relevant matrix equations can be written in the following form:

$$\begin{bmatrix} A_{N+j} \\ B_{N+j} \end{bmatrix} = \mathbf{R}^{(N+j)}\mathbf{R}^{(N+j-1)} \ldots \mathbf{R}^{(j)} \begin{bmatrix} A_j \\ B_j \end{bmatrix}$$

where an elementary unit in the multilayer consists of N different magnetic layers, i.e. the T-matrix is $\mathbf{R}^{(-)}\mathbf{R}^{(N-1)} \ldots \mathbf{R}^{(1)}$.

From the transformation properties of the matrix $\mathbf{R}^{(j)}$, we find that

$$\det \mathbf{T} = 1. \tag{1}$$

According to Bloch's law and (1), two eigenvalues of the T-matrix can be given as $e^{ik_\perp L}$ and $e^{-ik_\perp L}$. We can obtain two degenerate dispersion curves in the system through the standard calculation of the matrix trace without making use of the characteristic equation of the T-matrix.

But if the problem we discuss has four independent equations for boundary conditions, the transformation equation of the 4 × 4 T-matrix is given by

$$\begin{bmatrix} A_{N+j} \\ B_{N+j} \\ C_{N+j} \\ D_{N+j} \end{bmatrix} = \mathbf{T} \begin{bmatrix} A_j \\ B_j \\ C_j \\ D_j \end{bmatrix}.$$

Since the T-matrix has the same transformation properties as the 2 × 2 T-matrix, equation (1) is still satisfied by the T-matrix.

Using Bloch's law and (1), the four eigenvalues of the T-matrix are given as $\exp(ik_\perp^{(1)}L)$, $\exp(ik_\perp^{(2)}L)$, $\exp(ik_\perp^{(3)}L)$, $\exp(ik_\perp^{(4)}L)$ and

$$\sum_{i=1}^{4} k_\perp^{(i)} = 0.$$

It is worthwhile noticing that in general the 4 × 4 T-matrix is not similar to the 2 × 2 T-matrix, because using the 2 × 2 T-matrix we can directly obtain a dispersion relation by finding the trace of the matrix, and it is not necessary to solve its characteristic equation (the results for the former and the latter are just the same). For the 4 × 4 T-matrix we have to solve the characteristic equation in order to obtain the dispersion relation of the matrix and it is rather difficult to distinguish which homologous eigenvalue an individual dispersion curve, usually obtained from computer calculation, belongs to, because there is probably overlap. If a degenerate dispersion curve group (for instance, $k_\perp^{(1)}$ and $k_\perp^{(2)}$) has been obtained from the characteristic equation of the 4 × 4 matrix and $k_\perp^{(1)} = -k_\perp^{(2)}$, we can deduce the relationship of the other two eigenvalues in the T-matrix, i.e. $k_\perp^{(3)} = -k_\perp^{(4)}$, in accordance with (1), and we get the equation

$$2\cos k_\perp^{(1)}L + 2\cos k_\perp^{(3)}L = \sum_{i=1}^{4} T_{ii}. \tag{2}$$

As an example, let us consider the problem of the magnetic multilayer in rectangular metallic waveguides. The problem is given in the same geometry as in [13] and its elementary unit consists of N different magnetic layers with uniaxial crystal permeabilities $\tilde{\mu}^{(j)}$ ($j = 1, 2, \ldots, N$).

The retarded modes inside the metallic waveguide have to satisfy the Maxwell's equation

$$\begin{aligned} &\nabla^2 \boldsymbol{H}^{(j,m)} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{H}^{(j,m)}) - \frac{\varepsilon^{(j)}}{c^2}\tilde{\mu}^{(j)}\frac{\partial^2 \boldsymbol{H}^{(j,m)}}{\partial t^2} = 0 \\ &mL + \sum_{\sigma=1}^{j+1} d_\sigma < y \leqslant mL + \sum_{\sigma=1}^{j} d_\sigma \end{aligned} \tag{3}$$

where L is defined here as

$$L = \sum_{\sigma=1}^{N} d_\sigma$$

and d_σ ($\sigma = 1, 2, \ldots, N$) gives the thickness of each layer of an elementary unit.

Considering the boundary conditions of the metallic waveguide, we assume that the formal solutions are

$$H_x^{(j,m)} = \{A_x^{(j,m)} \exp[\alpha_j(y - mL)] + B_x^{(j,m)} \exp[-\alpha_j(y - mL)]\} \sin k_1 x \sin k_2 z \tag{4}$$

$$H_y^{(j,m)} = \{A_y^{(j,m)} \exp[\alpha_j(y - mL)] + B_y^{(j,m)} \exp[-\alpha_j(y - mL)]\} \times (\cos k_1 x \sin k_2 z + C^{(j,m)} \sin k_1 x \cos k_2 z) \tag{5}$$

$$H_z^{(j,m)} = \{A_z^{(j,m)} \exp[-\alpha_j(y - mL)]\} + B_z^{(j,m)} \exp[-\alpha_j(y - mL)]\} \sin k_1 x \sin k_2 z \tag{6}$$

where $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are wavevectors along the x direction and the z direction, respectively.

Substituting (4), (5) and (6) in the wave equation (3), we find that

$$(\alpha_j^2 - k_1^2 - k_2^2)[k_1^2 + k_2^2 - (\mu_0^{(j)}/\mu_{xx}^{(j)})\,\alpha_j^2 - \varepsilon^{(j)}\mu_0^{(j)}\,\omega^2/c^2] + \varepsilon^{(j)}(\omega^2/c^2)\mu_{\mathrm{V}}^{(j)}\,(k_1^2 + k_2^2 - \varepsilon^{(j)}\mu_0^{(j)}\,\omega^2/c^2 - \alpha_j^2\,\mu_0^{(j)}/\mu_{\mathrm{V}}^{(j)}) = 0$$

$$k_1 = m\pi/a \qquad k_2 = n\pi/a$$

where m and n are two positive integers that must not be equal to zero simultaneously, a and b are the lengths of the sides in the waveguide, $\mu_0^{(j)}$ $\mu_{xz}^{(j)}$ and $\mu_{xx}^{(j)}$ are the components of the permeability tensor, and $\mu_{\mathrm{V}}^{(j)}$ is the reduced permeability in the Voigt configuration ($\mu_{\mathrm{V}}^{(j)} = \mu_{xx}^{(j)} + \mu_{xz}^{(j)2}/\mu_{xx}^{(j)}$).

Imposing the usual boundary conditions and $\nabla \cdot \boldsymbol{B}^{(j)} = 0$, we obtain the following matrix equation:

$$\begin{bmatrix} A_x^{(N+1,m)} \\ B_x^{(N+1,m)} \\ A_z^{(N+1,m)} \\ B_z^{(N+1,m)} \end{bmatrix} = \mathbf{R}^{(N)}\mathbf{R}^{(N-1)}\cdots\mathbf{R}^{(1)} \begin{bmatrix} A_x^{(1,m)} \\ B_x^{(1,m)} \\ A_z^{(1,m)} \\ B_z^{(1,m)} \end{bmatrix}$$

where $\mathbf{R}^{(j)}$ is the 4×4 matrix with

$$R_{11}^{(j)} = \frac{r_j}{2\beta_{j+1}} + \frac{\sin^2\theta_{j+1}\Phi_j r_j}{2\beta_{j+1}\Phi_{j+1}} + \frac{\cos^2\theta_{j+1} r_j \Psi_j}{2\beta_{j+1}\Psi_{j+1}}$$

$$R_{12}^{(j)} = \frac{1}{2r_j\beta_{j+1}} - \frac{\sin^2\theta_{j+1}\Phi_j}{2r_j\beta_{j+1}\Phi_{j+1}} - \frac{\cos^2\theta_{j+1}\Psi_j}{2r_j\beta_{j+1}\Psi_{j+1}}$$

$$R_{13}^{(j)} = \frac{\sin^2\theta_{j+1}\Psi_j r_j}{2\beta_{j+1}\Phi_{j+1}} - \frac{\cos^2\theta_{j+1}\Phi_j r_j}{2\beta_{j+1}\Psi_{j+1}}$$

$$R_{14}^{(j)} = R_{13}^{(j)}\ (r_j \to -r_j^{-1}) \qquad R_{21}^{(j)} = R_{12}^{(j)} \begin{pmatrix} r_j \to -r_j^{-1} \\ \beta_{j+1} \to \beta_{j+1}^{-1} \end{pmatrix}$$

$$R_{22}^{(j)} = R_{11}^{(j)} \begin{pmatrix} r_j \to r_j^{-1} \\ \beta_{j+1} \to \beta_{j+1}^{-1} \end{pmatrix} \qquad R_{23}^{(j)} = R_{13}^{(j)} \ (\beta_{j+1} \to -\beta_{j+1}^{-1})$$

$$R_{24}^{(j)} = R_{13}^{(j)} \begin{pmatrix} r_j \to r_j^{-1} \\ \beta_{j+1} \to \beta_{j+1}^{-1} \end{pmatrix} \qquad R_{31}^{(j)} = R_{13}^{(j)} \ (r_j \to -r_j)$$

$$R_{32}^{(j)} = R_{13}^{(j)} \ (r_j \to r_j^{-1}) \qquad R_{33}^{(j)} = R_{11}^{(j)} \qquad R_{34}^{(j)} = R_{12}^{(j)}$$

$$R_{41}^{(j)} = R_{13}^{(j)} \ (\beta_{j+1} \to \beta_{j+1}^{-1}) \qquad R_{42}^{(j)} = R_{13}^{(j)} \begin{pmatrix} r_j \to r_j^{-1} \\ \beta_{j+1} \to \beta_{j+1}^{-1} \end{pmatrix}$$

$$R_{43}^{(j)} = R_{12}^{(j)} \begin{pmatrix} r_j \to r_j^{-1} \\ \beta_{j+1} \to \beta_{j+1}^{-1} \end{pmatrix} \qquad R_{44}^{(j)} = R_{11}^{(j)} \begin{pmatrix} r_j \to r_j^{-1} \\ \beta_{j+1} \to \beta_{j+1}^{-1} \end{pmatrix}$$

where

$$\sin\theta_j = \frac{\Phi_j}{\sqrt{(\Phi_j^2 + \Psi^2)}} \qquad \Phi_j = \frac{\mu_{xx}^{(j)}}{\alpha_j} \qquad \Psi = \frac{\mu_{xz}^{(j)}}{\alpha_j}$$

$$r_j = \exp\left(\alpha_j \sum_{\sigma=1}^{j} d_\sigma\right) \qquad \beta_{j+1} = \exp\left(\alpha_{j+1} \sum_{\sigma=1}^{j} d_\sigma\right)$$

and it is to be noted that

$$\mathbf{R}^{(N)} = \mathbf{R}^{(j)} \begin{pmatrix} j \to N \\ j+1 \to 1 \\ \beta_{j+1} \to 1 \end{pmatrix}.$$

The form of the T-matrix is completely defined by $\mathbf{T} = \mathbf{R}^{(N)}\mathbf{R}^{(N-1)} \ldots \mathbf{R}^{(1)}$.

It is theoretically feasible to obtain the coupling dispersion relation between the electromagnetic wave and spin-wave excitations in the system using the characteristic equation of the 4×4 T-matrix.

For the sake of simplicity, we take $N = 2$ and assume that one layer in an elementary unit is magnetic while the other is non-magnetic. With some algebraic manipulation, we obtain the dispersion relation

$$\begin{aligned}
&\sinh^2(\alpha_1 \mathrm{d}_1)\sinh^2(\alpha_2 d_2)[(\alpha_1^2\mu^2 + \alpha_2^2\mu_{xx}\mu_{\mathrm{V}})^2 - 4\alpha_2^2\mu_{xz}^2\alpha_1^2\mu^2] \\
&\quad + (\alpha_1^3\mu^3\alpha_2\mu_{xx} + \alpha_2^3\mu_{xx}^2\mu_{\mathrm{V}}\alpha_1\mu)\,[\sinh(2\alpha_1 d_1)\sinh(2\alpha_2 d_2) \\
&\quad - 4\sinh(\alpha_1 \mathrm{d})\sinh(\alpha_2 d_2)\cos(k_\perp L)] + \alpha_2^2\mu_{xx}\mu_{\mathrm{V}}\alpha_1^2\mu^2 \\
&\quad \times \{2\cos(2k_\perp L) + 4\cosh(\alpha_1 d_1)\cosh(\alpha_2 d_2)[\cosh(\alpha_1 d_1)\cosh(\alpha_2 d_2) \\
&\quad - 2\cos(k_\perp L)]\} = 0.
\end{aligned}$$

For an analysis of the above equation's solution, refer to [13].

From the dispersion relation obtained we can see that if $\mathrm{e}^{ik_\perp L}$ is a solution of the equation, then $\mathrm{e}^{-ik_\perp L}$ must also be a solution—that is, the equation has a group of degenerate dispersion curves—thus we can get the other group of degenerate curves from (2), i.e.

$$\cos(k_\perp^{(1)}L) + \cos(k_\perp^{(2)}L) = 2\cosh(\alpha_1 d_1)\cosh(\alpha_2 d_2) + \sinh(\alpha_1 d_1)\sinh(\alpha_2 d_2)[(\alpha_1/\alpha_2)\,\mu/\mu_V + (\alpha_2/\alpha_1)\,\mu_{xx}/\mu]$$

where μ is the permeability of the non-magnetic layers.

We assumed that the magnetic layers in the waveguide are all ferromagnetic and with the same physical parameters as given in [13]. We found through numerical calculation that the states at $k_\perp L = 1.2^{\eta_3}$ are two-fold degenerate and those where $k_\perp L$ is close to zero are all fourfold degenerate in figure 2 of [13].

We have discussed an improved T-matrix method for PPM and its application in the problem of a rectangular metallic waveguide. Although the method is restricted to problems that have four independent boundary condition equations and two groups of degenerate characteristic states (the two-sublattice model of antiferromagnets in the Heisenberg case belongs to this class too), most cases of 4×4 T-matrices fortunately do have two groups of degenerate states due to the symmetry of physical systems.

We wish to thank Dr J Barnaś for his enlightening work. This work was supported financially by the National Fund of Natural Sciences of China.

References

[1] Grimsditch M *et al* 1983 *Phys. Rev. Lett.* **51** 498
[2] Grünberg P and Mika K 1983 *Phys. Rev.* B **27** 2955
[3] Camley R E and Mills D L 1984 *Phys. Rev.* B **29** 1695
[4] Kueny A *et al* 1984 *Phys. Rev.* B **29** 2879
[5] Mika K and Grünberg P 1984 *Phys. Rev.* B **31** 4465
[6] Zhou Ching and Cao Shu 1985 *Chinese Phys. Lett.* **2** 385
[7] Hinchey L L and Mills D L 1986 *Phys. Rev.* B **33** 3329
[8] Dobrzynski L *et al* 1986 *Phys. Rev.* B **33** 3251
[9] Albuquerque E L *et al* 1986 *Solid State Commun.* **58** 41
[10] Zhu Ninjia and Cao Shu 1987 *Phys. Lett.* **124A** 515
[11] Barnaś J 1988 *J. Phys. C: Solid State Phys.* **21** 1021
[12] Barnaś J 1988 *J. Phys. C: Solid Phys.* **21** 4097
[13] Zhu Ninjia and Cao Shu 1989 *Phys. Lett* **134A** 264